

The Over-Extended Kalman Filter - Don't Use It!

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Abstract

Some target tracking filters ignore the radar range rate measurements because they are highly nonlinear in Cartesian space. A linearized measurement equation, composed of range rate's partial derivatives with respect to the track state, is sometimes used in an extended Kalman filter. Unfortunately, this naive linearization biases the posterior estimates. The origins of these biases are investigated and found to lie in the linearized measurement's functional dependence on the state's position elements. An alternative linearization having no functional dependence on the state's position elements is derived. Analyses and numerical evaluations show that the new linearization leads to filters with lower biases.

1 Introduction

Kalman filters are Bayes optimal, minimum mean squared error estimators for linear systems with Gaussian noises. Extended Kalman filters (EKF) replace nonlinear process and measurement equations (1) and (2)

$$\theta_{k+1} = f(\theta_k) + w_k \quad (1)$$

$$z_k = h(\theta_k) + v_k \quad (2)$$

with an approximate linear system composed of the partial derivatives of the nonlinear functions.

$$\theta_{k+1} = \left[\frac{\partial f(\theta)}{\partial \theta} \right]_{\theta=\theta_k} \theta_k + w_k \quad (3)$$

$$z_k = \left[\frac{\partial h(\theta)}{\partial \theta} \right]_{\theta=\theta_k} \theta_k + v_k \quad (4)$$

EKFs work well if the linearization errors are small and the system operates within a restricted range [9]. Their accuracy quickly degrades otherwise [17]. It is well known that naive linearizations can introduce biases or errors in the covariance calculations that degrade filter performance [2].

Range rate, also known as the radial or Doppler velocity, is highly nonlinear for a radar tracking problem set in Cartesian coordinates.

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{\sqrt{x^2 + y^2}} \quad (5)$$

Thus the radar range rate measurement is unsuited for a Kalman filter and many tracking systems do not use it to estimate the target's state. Nevertheless, it seems that proper use of the range rate measurement can improve a tracker, especially its velocity estimates. It is, after all, the only radar measurement that contains any information about target velocity.

Tracking systems with EKFs that linearize range rate as in (4) appear occasionally in the literature. Sometimes they work well [7]. Sometimes the authors reports that they diverge. Schutz *etal* [14] for example, state that an experimental EKF tracker designed for the E-2C airborne early warning aircraft diverges for most target trajectories in the sense that its position errors grow with time. The authors never explain why that happens, but later trackers for the same system [15], [16] do not diverge. The later trackers use a linearized range rate measurement in what the authors identify as a combined Kalman filter (CKF). A combined Kalman filter is a two stage, ad-hoc approach. In the first stage an ordinary Kalman filter processes the position measurements only. Then another Kalman filter processes the range rate measurement only. It appears that the authors learned that separating the range rate from the position measurements solved the problem without necessarily understanding why.

A small body of literature examining the stability of EKFs exists. Reif and Unbehauen [13] derive sufficient conditions that guarantee stochastic stability

of the EKF. The sufficient conditions take the form of constraints on the initial estimation errors and noise levels. In [11] they show that the EKF converges exponentially for nonlinear, deterministic, noise free systems.

Pearson *etal* [10] note that EKFs diverge, or convergence slowly, because the calculated covariance matrices become unrealistically small. Reif, Sonnemann and Unbehauen [12] modify the usual EKF equations by introducing an additive, exponentially weighted instability term to the process covariance. Instead of calculating the prior covariance $P^- = FPF^T + W$ where W is the process noise covariance, according to the KF recursion, they calculate $P^- = (F + \alpha I)P(F + \alpha I)^T + W$. For $\alpha = 0$ their filter is the EKF. For $\alpha > 0$ their filter is an EKF with a heuristically increased state covariance. Increasing the state covariance increases the filter's gain and sensitivity to measurements. They show that the new filter tolerates initial estimation errors better than the usual EKF, but is more sensitive to the initial covariance estimate. Boutayeb, Rafaralahy and Darouach [5] take a similar approach. They note that while letting W equal the process noise covariance is optimal of linear systems, it has never been proved optimal for nonlinear systems. They calculate prior state covariance as $P^- = FPF^T + W$, where W is not the process noise but an arbitrarily chosen matrix, and show that convergence improves when W is well chosen. These two approaches are identical when $W = W + \alpha(FP + PF^T) + \alpha^2P$. Boutayeb and Aubry [4] find weak conditions on the arbitrarily chosen noise covariance matrices that guarantee convergence.

Unfortunately, this work has little practical value for the target tracking problem. Reif and Unbehauen [13] conclude that the sufficient conditions for stochastic stability are so conservative that they have theoretical interest only. Real tracking systems are neither deterministic nor noise free, as assumed in [11]. In this paper, we show by example that even when the track system is deterministic and noise free, the EKF converges to the true state slowly. The authors of [4], [5] and [12] note that increasing the state covariance increases both the gain and the system's sensitivity to the initial covariance estimate. Increasing it too much leads to an oscillating estimator. They offer no guidance for choosing the covariance, other than suggesting extensive simulations and ensuring that the resulting matrices are nonsingular.

Kosuge *etal* [8] present a novel tracker that attacks the problem at its source. The authors notice that when the target lies exactly on one of the coordinate axes the range rate is a linear function of the target's speed in the direction of the axis. They rotate the tracking frame of reference for each target so that its estimated position is exactly on one axis. Then the measurement consists of linear position measurements and one linear velocity measurement. The state is updated using an ordinary KF. Additional computations, calculation of the 4x4 rotation matrices, rotating prior to updating and rotating back to the original frame, are necessary.

Reports that EKF trackers diverge [14] are not surprising given the EKF's sensitivity to linearization errors and covariance calculations. However, it is

surprising that the specific cause of divergence is not more widely examined.

In this paper, we investigate the divergence phenomenon and propose an alternative range rate measurement linearization. We compare an ordinary KF using only the position measurements and two EKFs, one using the usual and the other the alternative linearization, analytically and via simulation. The analytical results show that the usual linearization introduces unreasonable biases in the posterior estimates. The alternative linearization improves the velocity estimates without introducing unreasonable gains on the positions. Numerical examples show significant posterior biases in the usual EKF for realistic covariances and innovations. The usual EKF is so sensitive to small changes in the state estimates that we call it the over-extended Kalman filter (OEKF). We show that the CKF induces unreasonable biases and can be improved with the alternative linearization. Furthermore, an EKF with the alternative linearization is nearly identical to Kosuge et al's coordinate transformation tracker.

We observe and briefly discuss three paradoxes arising from the usual linearization. First, under certain conditions, the OEKF's posterior position and range rate estimates grow insensitive to their innovations. The position innovations updates the range rate estimate and the range rate innovation updates the position estimate. Second, the usual linearization leads to filters whose net effective gains can be outside $[0, 1]$. Finally, increasing the strength of your prior beliefs by decreasing the state variance can actually increase the sensitivity of the posterior estimate to the innovation. We explain the paradoxes as natural consequences of the linearization.

2 Alternative Linearization

Consider a two dimensional radar at the origin measuring range ρ , bearing γ , and range rate \dot{r} . Two dimensions suffice for demonstrating the differences between the ordinary and alternative linearization. It is easy to extend the problem to radar located elsewhere or to three dimensions. The track state consists of a vector of positions and velocities (6).

$$\theta = \begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix}^T \tag{6}$$

Range and bearing are converted to Cartesian coordinates (7) in the usual way [2],

$$\begin{bmatrix} x \\ y \end{bmatrix} = \rho \begin{bmatrix} \sin \gamma \\ \cos \gamma \end{bmatrix} \tag{7}$$

so the measurements used by the tracking system are

$$z = \begin{bmatrix} x \\ y \\ \dot{r} \end{bmatrix} \tag{8}$$

Range rate (9) is nonlinear in this system.

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{\sqrt{x^2 + y^2}} \quad (9)$$

It can be linearized as in (4). An extended Kalman filter with process and measurement equations (10) and (11), respectively,

$$\theta_k = F\theta_{k-1} + w_k \quad (10)$$

$$z_k = H_k\theta_k + v_k \quad (11)$$

estimates the track state. F is linear motion. w_k represents a white noise acceleration of constant magnitude during the interval between measurement. Its variance is W_k , where

$$W_k = \begin{bmatrix} \frac{1}{2}\Delta t_k^2 & 0 \\ 0 & \frac{1}{2}\Delta t_k^2 \\ \Delta t_k & 0 \\ 0 & \Delta t_k \end{bmatrix} \begin{bmatrix} \sigma_x^2 \\ \sigma_y^2 \end{bmatrix}_k \begin{bmatrix} \frac{1}{2}\Delta t_k^2 & 0 \\ 0 & \frac{1}{2}\Delta t_k^2 \\ \Delta t_k & 0 \\ 0 & \Delta t_k \end{bmatrix}^T \quad (12)$$

v_k is white measurement noise with covariance V_k , and

$$H_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\partial \dot{r}}{\partial x} & \frac{\partial \dot{r}}{\partial y} & \frac{\partial \dot{r}}{\partial \dot{x}} & \frac{\partial \dot{r}}{\partial \dot{y}} \end{bmatrix} \quad (13)$$

with the last row evaluated at the current state estimate. This is the usual extended Kalman filter for radar target tracking [7].

Writing the partial derivatives explicitly, multiplying by the state vector, and combining terms makes it clear that the expected range rate measurement equals the true range rate.

$$\begin{aligned} E[\dot{r}] &= x_k \left(\frac{\dot{x}_k}{\sqrt{x_k^2 + y_k^2}} - \frac{x_k(x_k\dot{x}_k + y_k\dot{y}_k)}{(x_k^2 + y_k^2)^{3/2}} \right) + \\ & y_k \left(\frac{\dot{y}_k}{\sqrt{x_k^2 + y_k^2}} - \frac{y_k(x_k\dot{x}_k + y_k\dot{y}_k)}{(x_k^2 + y_k^2)^{3/2}} \right) + \\ & \dot{x}_k \left(\frac{x_k}{\sqrt{x_k^2 + y_k^2}} \right) + \\ & \dot{y}_k \left(\frac{y_k}{\sqrt{x_k^2 + y_k^2}} \right) \end{aligned} \quad (14)$$

$$E[\dot{r}_k^{(m)}] = \frac{x_k\dot{x}_k + y_k\dot{y}_k}{\sqrt{x_k^2 + y_k^2}} \quad (15)$$

The first two elements of (14) sum to zero. Multiplying through yields

$$\frac{x_k\dot{x}_k}{\sqrt{x_k^2 + y_k^2}} - \frac{x_k^2(x_k\dot{x}_k + y_k\dot{y}_k)}{(x_k^2 + y_k^2)^{3/2}} + \frac{y_k\dot{y}_k}{\sqrt{x_k^2 + y_k^2}} - \frac{y_k^2(x_k\dot{x}_k + y_k\dot{y}_k)}{(x_k^2 + y_k^2)^{3/2}} \quad (16)$$

Combining the first and third, and the second and fourth terms together

$$\frac{x_k \dot{x}_k + y_k \dot{y}_k}{\sqrt{x_k^2 + y_k^2}} - \frac{(x_k^2 + y_k^2)(x_k \dot{x}_k + y_k \dot{y}_k)}{(x_k^2 + y_k^2)^{3/2}} \quad (17)$$

Factoring $(x_k^2 + y_k^2)$ out of the second term yields

$$\frac{x_k \dot{x}_k + y_k \dot{y}_k}{\sqrt{x_k^2 + y_k^2}} - \frac{(x_k \dot{x}_k + y_k \dot{y}_k)}{\sqrt{x_k^2 + y_k^2}} = 0 \quad (18)$$

Thus, an alternative linearization and a filter with measurement matrix (19) exist.

$$H_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\partial \dot{r}}{\partial \dot{x}} & \frac{\partial \dot{r}}{\partial \dot{y}} \end{bmatrix} \quad (19)$$

We call the EKF using (19) the alternative extended Kalman filter (AEKF). In section 3, we explore the differences between a KF processing only the position measurements, the OEKF and the AEKF. We pay special attention to the gains and find the OEKF's gains unreasonable. Systems with uncorrelated and correlated noises, and a deterministic noise-free system, are analyzed. We also show that the alternative linearization improves the CKF. Examples in section 4 demonstrate the results of section 3.

3 Filter Differences

The differences in the third rows of (13) and (19) significantly impact filter gain. We demonstrated this by solving the recursive Kalman filter equations for each filter. To simplify the presentation, let a, b, c, d denote the partial derivatives $\frac{\partial \dot{r}_k}{\partial x_k}, \frac{\partial \dot{r}_k}{\partial y_k}, \frac{\partial \dot{r}_k}{\partial \dot{x}_k}, \frac{\partial \dot{r}_k}{\partial \dot{y}_k}$, let P^- denote the 4x4 prior state covariance matrix, and let H_0, H_{oekf} , and H_{alt} denote the measurement matrices of the KF, OEKF, and AEKF, respectively,

$$H_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (20)$$

$$H_{oekf} = \begin{bmatrix} H_0 \\ a & b & c & d \end{bmatrix} \quad (21)$$

$$H_{alt} = \begin{bmatrix} H_0 \\ 0 & 0 & c & d \end{bmatrix} \quad (22)$$

The OEKF's range rate measurement is functionally dependent upon the state's position elements. The AEKF's depends only on the velocity elements. This difference leads to important, sometimes counterintuitive, differences in the filter gains.

Measurement covariance S , just before taking the measurement, equals $HPH^T + V$. The 2x2 matrix S_0 for the KF is

$$S_0 = H_0 P H_0^T + \begin{bmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{bmatrix} \quad (23)$$

The upper left elements of the 3x3 matrices S_{oekf} and S_{alt} equal S_0 . Their third columns, and the corresponding elements of their third rows, are

$$S_{oekf}(:, 3) = \begin{bmatrix} [abcd] P_{.1}^- \\ [abcd] P_{.2}^- \\ [abcd] P^- [abcd]^T + v_{33} \end{bmatrix} \quad (24)$$

$$S_{alt}(:, 3) = \begin{bmatrix} [00cd] P_{.1}^- \\ [00cd] P_{.2}^- \\ [00cd] P^- [00bcd]^T + v_{33} \end{bmatrix} \quad (25)$$

where $P_{.j}^-$ is the j^{th} column of P^- .

3.1 Gains In the General Case

Expressions for the gain matrices, $K = PH^T S^{-1}$, as functions of the matrix elements are easy to obtain but algebraically cumbersome. Letting $\{k\}_{ij}$ denote the elements of K_0 , $\{\alpha\}_{ij}$ and $\{\epsilon\}_{ij}$ denote the elements of K_{alt} and K_{oekf} , respectively,

$$K_0 = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \\ k_{31} & k_{32} \\ k_{41} & k_{42} \end{bmatrix} \quad (26)$$

$$K_{oekf} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \\ \epsilon_{41} & \epsilon_{42} & \epsilon_{43} \end{bmatrix} \quad (27)$$

$$K_{alt} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} \end{bmatrix} \quad (28)$$

The posterior state estimate, θ_{post} , equals the prior estimate, θ_{prior} , plus the measurement innovations premultiplied by the gain (29). The first two columns of K are gains on the position innovations, the third column the range rate innovation. Of course, K_0 lacks a third column because it only uses the position measurements.

$$\theta_{post} = \theta_{prior} + K(z - H\theta_{prior}) \quad (29)$$

In the general case all three filters have different gains. The ϵ 's and α 's are nonzero and difficult to interpret directly. In the following section we make

a simplifying assumptions that eases their interpretation and highlights their differences.

3.2 Gains In an Uncorrelated System

Kalman filters are linear. They are invariant under scale and rotation and the coordinate system can be chosen arbitrarily. A coordinate system that simplifies the analysis and interpretation of results places the prior track state estimate exactly on the y -axis at a range of 1. In this system \dot{y} equals the range rate. The prior state vector is $\theta_{prior} = [0 \ 1 \ \dot{x} \ \dot{y}]^T$ and the partial derivatives are

$$\begin{bmatrix} \frac{\partial \dot{r}}{\partial x} & \frac{\partial \dot{r}}{\partial y} & \frac{\partial \dot{r}}{\partial \dot{x}} & \frac{\partial \dot{r}}{\partial \dot{y}} \end{bmatrix} = \begin{bmatrix} \dot{x} & 0 & 0 & 1 \end{bmatrix} \quad (30)$$

Measurements x , y , and \dot{r} are also measurements of cross-radial position, radial position, and range rate.

Assume that the prior state and measurement vectors are uncorrelated; i.e., P^- and V are diagonal matrices. This assumption is unrealistic in practice because the acceleration noises (12) always introduce a positive correlation between the positions and velocities. Nevertheless, it highlights the differences between the KF, OEKF and AEKF.

The gains are

$$K_0 = \begin{bmatrix} \frac{p_{11}}{(p_{11}+v_{11})} & 0 \\ 0 & \frac{p_{22}}{(p_{22}+v_{22})} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (31)$$

$$K_{oekf} = \begin{bmatrix} \epsilon_{11} & 0 & \epsilon_{13} \\ 0 & \frac{p_{22}}{(p_{22}+v_{22})} & 0 \\ 0 & 0 & 0 \\ \epsilon_{41} & 0 & \epsilon_{43} \end{bmatrix} \quad (32)$$

$$K_{alt} = \begin{bmatrix} \frac{p_{11}}{(p_{11}+v_{11})} & 0 & 0 \\ 0 & \frac{p_{22}}{(p_{22}+v_{22})} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{p_{44}}{(p_{44}+v_{33})} \end{bmatrix} \quad (33)$$

The KF and AEKF have identical position gains equaling the ratio of prior state to prior measurement variance. The AEKF range rate gain, or the gain on \dot{y} , also equals the ratio of prior state to prior measurement variance. This is an intuitively satisfying situation for the uncorrelated system. The position innovations update the position estimates, the range rate innovation updates the range rate estimate, and the gains are ratios of variances. The gain on \dot{y} in (33) effects the results of the coordinate transformation in [8] without rotating the system.

The OEKF has different cross-radial position and range rate gains that the AEKF. Notice that both cross-radial position and range rate innovations update the cross-radial position and range rate estimates. That is, a measurement

containing only position information updates the velocity estimate and a measurement containing only velocity information updates the position estimate. This oddity is a consequence of nonzero a and b in (21). It is not a consequence of correlations because the system is totally uncorrelated.

Writing the OEKF gains explicitly,

$$\epsilon_{11} = \frac{p_{11} (p_{44} + v_{33})}{p_{11} (p_{44} + v_{33} + \dot{x}^2 v_{11}) + v_{11} (p_{44} + v_{33})} \quad (34)$$

$$\epsilon_{13} = \frac{\dot{x} p_{11} v_{11}}{p_{11} (p_{44} + v_{33} + \dot{x}^2 v_{11}) + v_{11} (p_{44} + v_{33})} \quad (35)$$

$$\epsilon_{41} = \frac{-\dot{x} p_{11} p_{44}}{p_{11} (p_{44} + v_{33} + \dot{x}^2 v_{11}) + v_{11} (p_{44} + v_{33})} \quad (36)$$

$$\epsilon_{43} = \frac{p_{44} (p_{11} + v_{11})}{p_{11} (p_{44} + v_{33} + \dot{x}^2 v_{11}) + v_{11} (p_{44} + v_{33})} \quad (37)$$

we see that they are not easy to interpret. Their common denominator

$$p_{11} (p_{44} + v_{33} + \dot{x}^2 v_{11}) + v_{11} (p_{44} + v_{33}) \quad (38)$$

increases linearly with the square of \dot{x} . The cross term gains, ϵ_{13} and ϵ_{41} , are linear in \dot{x} , with negative and positive slopes, respectively. All gains are functions of the product of a position and a velocity variance.

As the prior heading estimate approaches a radial direction, \dot{x} approaches zero and the common denominator approaches $(p_{11} + v_{11})(p_{44} + v_{33})$. Terms cancel in the numerators and denominators of ϵ_{11} and ϵ_{43} and they approach α_{11} and α_{43} . The cross term gains ϵ_{13} and ϵ_{41} approach zero. K_{oekf} converges to K_{alt} . The alternative and over extended Kalman filters produce identical solutions in an uncorrelated system when the prior heading is exactly radial, either toward or away from the radar.

As the prior heading estimate approaches a cross-radial direction, \dot{x} grows large. ϵ_{11} and ϵ_{43} decrease and reach their minima when the heading is exactly cross-radial. The cross term gains ϵ_{13} and ϵ_{41} increase at first then decrease. But, they decrease slowly because their numerators grow linearly with \dot{x} . Thus we have observe an interesting and wholly counterintuitive paradox. *As the prior heading becomes cross radial, the posterior position and range rate estimates grow insensitive to their respective innovations.* ϵ_{11} can become so small relative to ϵ_{13} that the posterior position estimate depends more on the range rate innovation than on the position innovation.

It is reasonable to expect the posterior position to lie between the prior and measurement. That is, the net effective gain should be in $[0, 1]$. The OEKF lacks this property. This is the second paradox of the usual linearization: *it leads to filters whose posterior estimates can move from the prior, toward and beyond the*

measurement. Or, they can move from the prior away from the measurement. This is shown by example later.

3.3 Combined Kalman Filter

Schutz *etal* report that an OEKF diverges for most target trajectories [14] but never explain why. Later, they describe a two stage filter called combined Kalman filter tracker [15], [16]. It processes on the position measurements in the first stage, and then the range rate measurement. They claim that this tracker does not diverge. It appears that the authors learned that separating the position and range rate measurements improves the filter, but they might not understand why. In this section we compare the combined Kalman filter using the usual and alternative linearizations.

In its first stage, the CKF is an ordinary KF. Its second stage has process and measurement equations

$$\theta_k = \theta_k \quad (39)$$

$$z_k = H_k \theta_k + v_k \quad (40)$$

where

$$H_k = \begin{bmatrix} \frac{\partial \dot{r}}{\partial x} & \frac{\partial \dot{r}}{\partial y} & \frac{\partial \dot{r}}{\partial \dot{x}} & \frac{\partial \dot{r}}{\partial \dot{y}} \end{bmatrix} \quad (41)$$

Denoting the elements of \tilde{H} by a , b , c and d as before, the combined KF's gain \tilde{K}_{oekf} is

$$\tilde{K}_{oekf} = \begin{bmatrix} [abcd] P_1 \\ [abcd] P_2 \\ [abcd] P_3 \\ [abcd] P_4 \end{bmatrix} / \left([abcd] P [abcd]^T + v_{33} \right) \quad (42)$$

If the alternative linearization is used, the gain \tilde{K}_{alt} is easily found from (42) by replacing a and b with zero. In the uncorrelated case, the gains are

$$\tilde{K}_{oekf} = \begin{bmatrix} \frac{\dot{x} p_{11}}{x^2 p_{11} + p_{44} + v_{33}} \\ 0 \\ 0 \\ \frac{p_{44}}{x^2 p_{11} + p_{44} + v_{33}} \end{bmatrix} \quad (43)$$

$$\tilde{K}_{alt} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{p_{44}}{p_{44} + v_{33}} \end{bmatrix} \quad (44)$$

The CKF should be better than the OEKF because its always has reasonable first stage position gains. Unlike the OEKF, the CKF's posterior position is always sensitive to position innovation. But unless the prior heading estimate

is exactly radial it is also sensitive to range rate innovation. If the alternative linearization is used then the CKF's posterior position estimate does not depend on the range rate innovation.

4 Examples

This section has four examples comparing the OEKF and AEKF posterior position and velocity estimates in an uncorrelated system. We discuss the effects of parameter changes on the OEKF's posterior position biases. An example with a correlated systems shows that the OEKF can have nontrivial biases in some realistic cases. Finally, an example of a deterministic noise-free system shows that even though the OEKF converges, it does so slowly.

When filtering, it is reasonable to expect the posterior estimate to move from the prior toward the measurement and lie somewhere in between. That it, its gains should be in the interval $[0, 1]$. It is easy to show via simulation that the OEKF's position estimates lack this property for all prior headings. The posterior can move toward the measurement, but beyond it. Or, it can move in the opposite direction. Even if the position measurement and prior are near in statistical distance, the posterior can be several standard deviations away from both. The net effect can be interpreted as a gain that lies outside $[0, 1]$.

4.1 Example 1

Figures 1 and 5 AEKF and OEKF posterior position and velocity estimates for the uncorrelated system. The track's prior estimated position always lies exactly on the y -axis at a range of 50 miles. Its prior estimated speed always equals 500 knots. State and measurement covariances are $P = \text{diag}([2, 2, 50, 50])$ and $V = ([1, 1, 5])$, and the range rate innovation is 20 knots. In figure 1 the large filled dot in the center is the prior position and the line indicates that heading is in the y direction, radially away from a radar at the origin. Eight measurements and the corresponding posterior estimated positions of the AEKF and OEKF are show. Measurements are shown every 45 degrees relative the prior.

Figure 5 shows the difference between the prior and posterior range rate estimates as a function of prior heading. Notice that the AEKF's estimate is the same for all headings; its gain depends only on the variance ratios. The OEKF's estimate varies with heading; its gain is a function of the variances and the prior x velocity and exhibits a pronounce asymmetry.

4.2 Example 2

The estimates in figure 2 have the same prior position and speed, state and measurement covariances, and range rate innovation as figure 1, but the prior heading is in the x direction, tangential to a radar at the origin. The AEKF's posterior position estimates are the same, as expected from 33. The OEKF's are biased in the positive x direction, as expected from 35. Some of the net

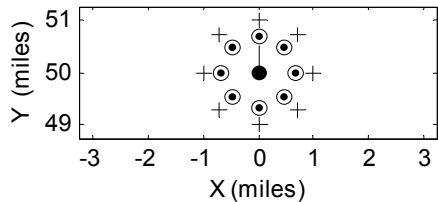


Figure 1: + Msmt, O AEKF, • OEKF Position comparison: baseline case

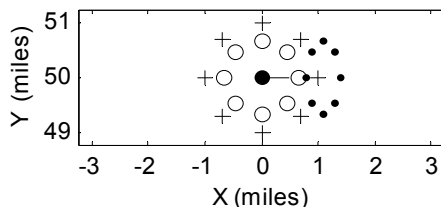


Figure 2: + Msmt, O AEKF, • OEKF Position comparison: off radial prior heading and large range rate innovation.

position gains lie outside $[0, 1]$. For example, the measurement at $(-1, 50)$ induces a posterior at about $(0.8, 50)$. The posterior position moved away from the measurement rather than toward it.

4.3 Example 3

Figures 3 and 6 show the effect of decreasing prior velocity variance. All parameters are the same as in example 1 except that $P = \text{diag}([2, 2, 10, 10])$. In figure 3, the prior heading is 30 degrees. As before, the effective gains are outside $[0, 1]$. The measurement at $(-1, 50)$ induce a posterior at about $(1.8, 50)$. The effective gain is negative, and the posterior is almost three standard deviations from the measurement. Figure 6 shows how the OEKF's posterior range rate varies with prior heading.

4.4 Example 4

Figure 4 shows the effect of increasing range rate innovation. All parameters are the same as in example 1 except that range rate innovation equals 50 knots and the prior heading is 75 degrees. The posterior position estimates are more than three standard deviations away from the measurements, and the effective gains are outside $[0, 1]$.

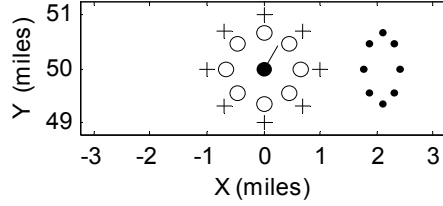


Figure 3: + Msmt, O AEKF, • OEKF
 Position comparison: off radial prior heading, and small prior velocity variance.

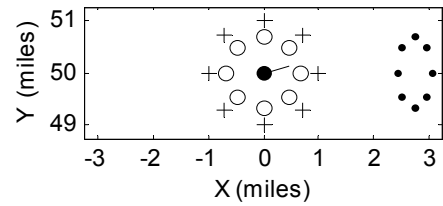


Figure 4: + Msmt, O AEKF, • OEKF
 Position comparison: off radial prior heading and large range rate innovation.

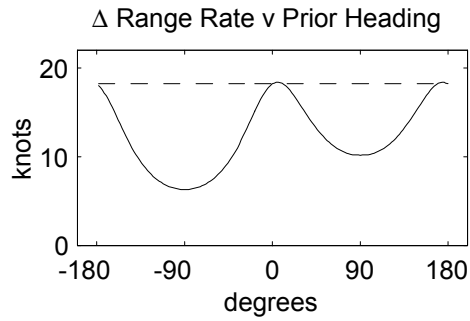


Figure 5: - - - AEKF, — OEKF Velocity comparison: baseline case

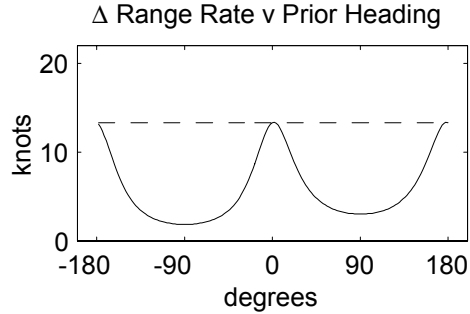


Figure 6: — — — AEKF, — OEKF
Velocity comparison: small prior velocity variance

The difference between the OEKF’s posterior velocity estimates and those of the other filters are minor. Posterior \dot{x} estimates are identical, as expected given the gains (31) to (32). Posterior \dot{y} estimates differ by no more than one half the range rate innovation for the examples in figures 1 to 4. In all cases, the net effective range rate gain is in $(0, 1)$.

The examples in this section show that the OEKF’s biases are not trivial. They increase when target heading is off-radial, cross radial position variance is large, the radial velocity variance is small, and the range rate innovation is large. In practice, these conditions arise when the radar’s bearing error is much greater than its range error, and a target that has been moving at a constant velocity for a long time suddenly maneuvers when its trajectory is off radial with respect to radar position. Divergence is possible with the OEKF, but only likely under these conditions. However, even though divergence is infrequent, it can be avoided altogether with the alternate linearization.

4.5 Example Of a Realistic, Correlated System

The covariance assumptions in the previous section are useful for highlighting the OEKF’s posterior biases, but unrealistic. In this section a more realistic covariance is determined via simulation and an example showing the worst case biases is presented. In the general case AEKF position gains do not identically equal to those of the KF because of non-zero prior state covariance.

More realistic covariances for a target at $(x, y) = (0, 50)$ are determined via simulation. The simulated target travels 500 miles per hour for a long time in the positive x direction along the line defined by $y = 50$ miles. The radar scans once every 10 seconds and has range, bearing, and range rate standard errors of 100 feet, 0.6 degrees and 2.5 knots, respectively. One simulation run produced the covariance matrix (45). (46) is the covariance of a measurement

having range 50 miles and bearing 0 degrees.

$$P = \begin{bmatrix} 0.04 & 0.0008 & 2.09 & 0.278 \\ 0.0008 & 0.001 & 0.057 & 0.215 \\ 2.09 & 0.057 & 290.0 & -8.48 \\ 0.278 & 0.148 & -8.48 & 67.0 \end{bmatrix} \quad (45)$$

$$V = \begin{bmatrix} 6.75 & 0 & 0 \\ 0 & .0004 & 0 \\ 0 & 0 & 5.25 \end{bmatrix} \quad (46)$$

If the target continues along its path at constant velocity then the simulated range rate innovations are almost always less than 20 knots. If the target makes a $2g$ coordinated turn its range rate can change by as much as 180 knots between scans. Posterior position estimates given prior state estimate $\theta_{prior} = [0 \ 50 \ 500 \ 0]^T$, covariance (45), and range rate innovations of 20 and 180 knots are shown in figures 7 and 8, respectively.

Figure 7 shows that small range rate innovations, such as those that can be expected for a nonmaneuvering target, produce significant OEKF position biases. Some of the net gains are seen to be outside $[0, 1]$. The OEKF's posterior y estimates nearly equal the AEKF's. Its posterior x estimates differ by nearly two miles, or about two-thirds the x -measurement standard deviation.

Figure 8 shows vary large OEKF position when the range rate innovation is large. In this case, the range rate innovation is 180 knots, such as might be observed when the target is maneuvering. As before, the posterior y estimates are nearly equal. The posterior x estimates can differ nearly three miles. This is only about one x -measurement standard deviation, but it is also about fifteen prior x estimate standard deviations. All net effective gains are seen to be outside $[0, 1]$.

Figures 9 and 10 show the change in the \dot{y} estimate as a function of prior target heading. The OEKF is highly sensitive to prior heading, but the AEKF is insensitive. This result is expected. The OEKFs range rate gains in equations (36) and (37) are complicated functions of \dot{x} . \dot{x} changes with the prior heading and so do the gains.

Figure 11 shows how slowly the OEKF converges for the deterministic, noiseless system with prior mean $\theta_0 = [0 \ 50 \ 500 \ 0]^T$, prior variance given by (45) when the true state is stable at $\theta = [0 \ 51 \ 500 \ 50]^T$. Measurements are noiseless and exactly equal the true state; they are all identical. Many measurements are taken and filtered sequentially. The AEKF takes a few iterations to converge. The OEKF takes hundreds.

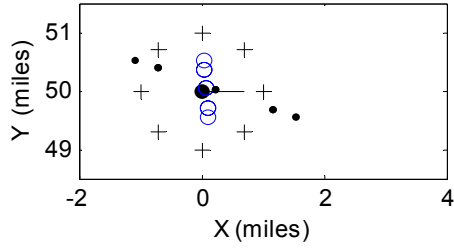


Figure 7: + Msmt, O AEKF, • OEKF
Position comparison: realistic covariances and cross radial prior heading.

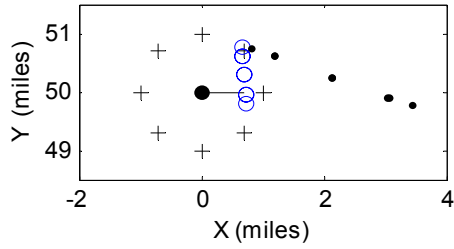


Figure 8: + Msmt, O AEKF, • OEKF
Position comparison: realistic covariances and cross radial prior heading.

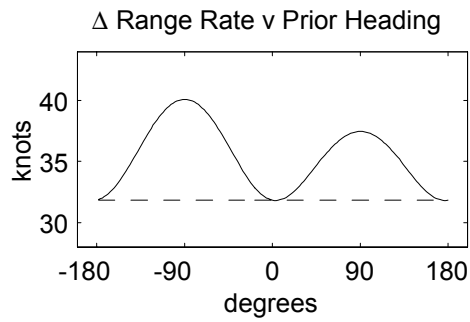


Figure 9: - - - AEKF, — OEKF
Velocity comparison: realistic covariances, range rate innovation = 20 knots

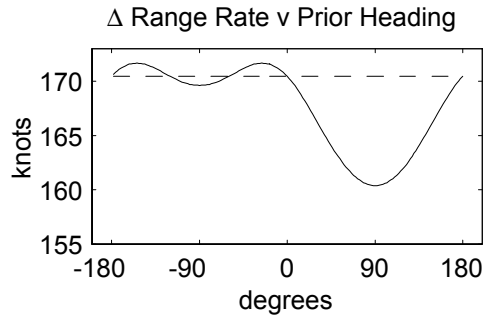


Figure 10: --- AEKF, — OEKF
 Velocity comparison: realistic covariances, range rate innovation = 180 knots

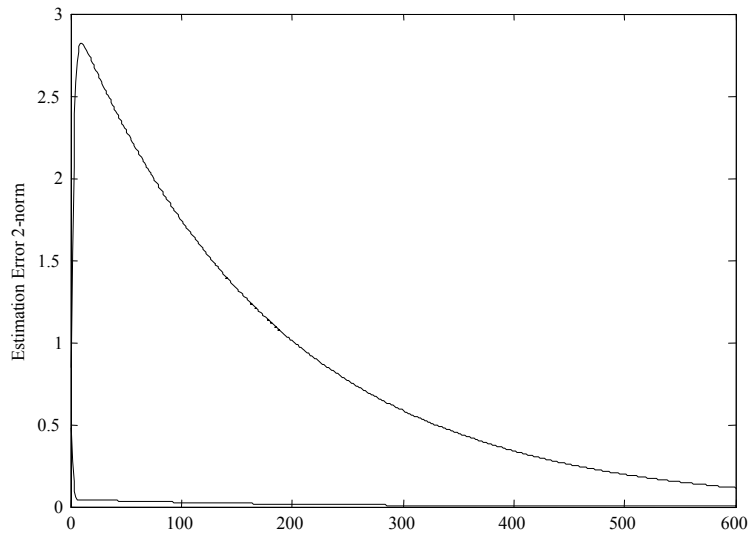


Figure 11: --- AEKF, — OEKF
 Position estimation error in a time invariant, noiseless system.

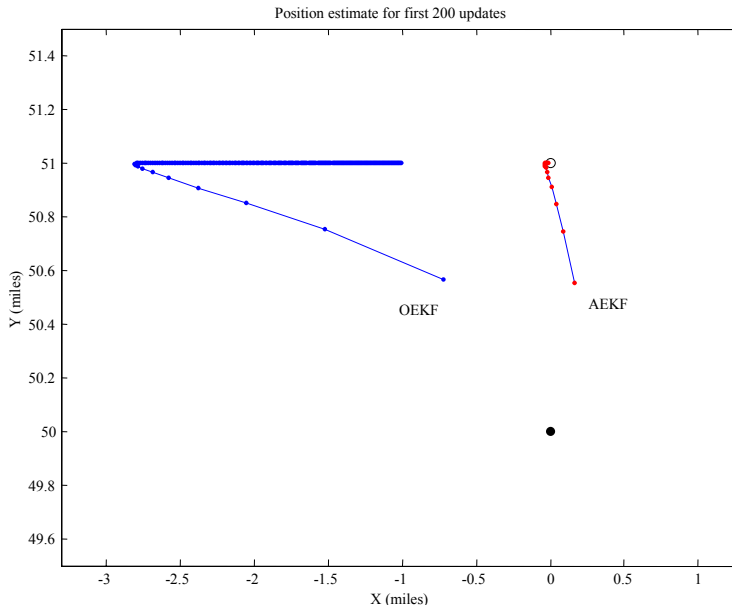


Figure 12: Sequential position estimates, first 200 measurements.

4.6 Example Of a Deterministic System

In this section the AEKF and OEKF are compared for a time invariant, deterministic, noise free system represented by

$$\theta_k = \theta \quad (47)$$

$$z_k = h_k(\theta_k) \quad (48)$$

Notice that the state never changes and that all measurements are identical. Reif and Unbehauen's [11] main result is quoted to show that the OEKF is an exponential observer for this system. However, even though the OEKF's solution converges exponentially, its rate is unacceptably slow. This is shown by example.

Definition 1 An estimator with errors ζ_n is said to have an exponentially stable equilibrium point at ζ if there are positive real numbers $\epsilon, \eta > 0$ and $\lambda > 1$ such that $\|\zeta_n\| \leq \eta \|\zeta_0\| \lambda^{-n}$ holds for every $n > 0$.

Definition 2 An estimator is said to be an exponential observer if it has an exponentially stable equilibrium point at ζ .

The main result of [11] follows.

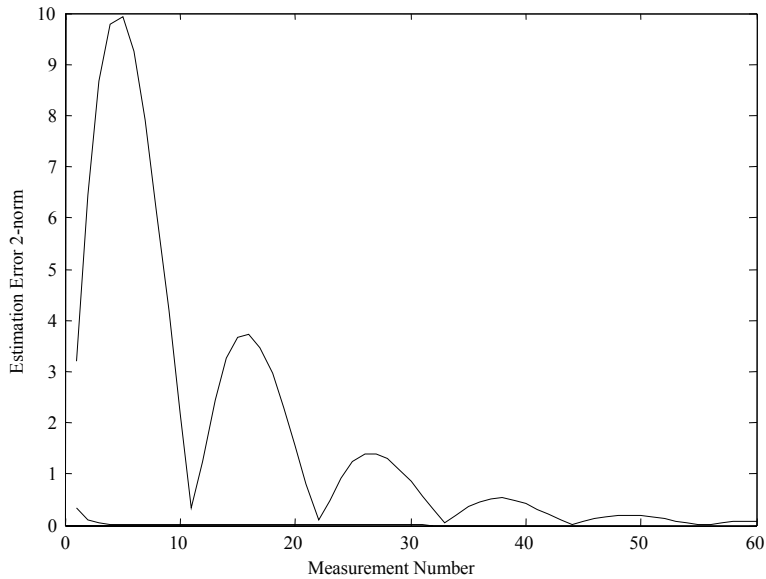


Figure 13: — — — AEKF, — OEKF
Position estimation error in a time invariant, noiseless system.

Theorem 3 Consider a discrete time extended Kalman filter for the system represented by (47) and (48). If there is a positive real number \bar{c} such that $\|\frac{\partial h}{\partial \theta}\| \leq \bar{c}$, the system is uniformly observable, and the linearization errors are bounded, then the extended Kalman filter is an exponential observer.

The previous theorem states that the OEKF is an exponential observer for the system. It does state the rate at which the OEKF’s solution converges. The following example shows that the OEKF converges much slower than the AEKF. In the example the system is time invariant, deterministic and noiseless. The initial prior state estimate is Gaussian with mean $\theta_0 = [0 \ 50 \ 500 \ 0]^T$ and variance $P_0 = \text{diag}([2, 2, 10, 10])$. The true state is stable at $\theta = [0 \ 51 \ 500 \ 50]^T$. Measurements are noiseless and exactly equal the true state; they are all identical. Many measurements are taken and filtered sequentially. Figure 13 shows the Euclidean position estimation errors of the AEKF and OEKF. The AEKF converges after a few iterations. The OEKF oscillates around the true state, and converges only after dozens of iterations.

5 OEKF Position Biases

The sensitivity of the OEKF’s posterior position estimates to P^- , V , θ_{prior} and range rate innovation was evaluated numerically using the previous and other

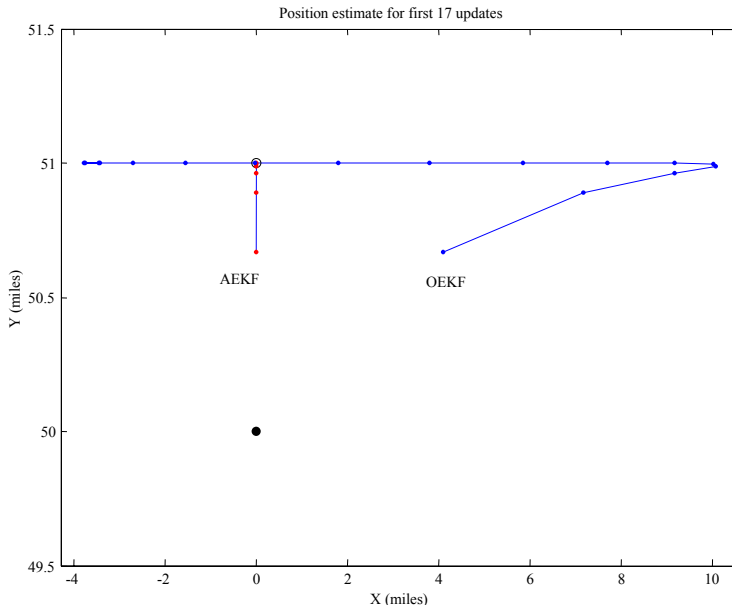


Figure 14: Sequential position estimates, first 17 measurements.

examples. Table 1 shows their effect on the OEKF’s biases.

When the prior velocity variances are large and the range rate innovation is small then the biases are moderate, as expected. It is well known that the EKF works well if the linearization errors are small and the initial estimate is good [10]. When the prior position variances are high, the position biases increase. OEKF position biases are sensitive to target heading. They are larger for off-radial headings than for radial headings, as stated earlier.

However, the range rate innovation can be substantially larger than its measurement’s standard deviation. Range rate is hard to estimate well because the estimates of all four state elements, position and velocity, must be accurate. Even if the position estimates are good, the velocity estimates may differ substantially from their true values. This is especially true when the target is maneuvering. Furthermore, the covariance matrix W_k in (12) produces a positive correlation between the x and \dot{x} errors, and the y and \dot{y} errors. Positive correlations tend to increase the estimation error, as can be seen from (14). Errors in the measurement and association processes also cause large range rate innovations. If the radar reports the range rate from one of the target’s multipath returns, or if the associator errs, then the range rate innovation will be high.

An interesting characteristic of the OEKF is that its position biases increase as its prior state velocity variance decreases. This is the third paradox: *stronger prior beliefs increase the biases in the posterior estimate*. This paradox

Parameter	Parameter change	Posterior Position Difference
p_{11}	↑	↑
p_{44}	↓	↑
v_{11}	↑	↑
v_{33}	↓	↑
range rate innovation	↑	↑
heading	cross radial	↑

Table 1: Relation between covariances, measurements, and OEKF posterior position biases

is a direct result of the functional relation between range rate and the position estimates in H_{oekf} . Low prior velocity variance reduces the sensitivity of the posterior velocity estimates to the range rate innovation, so the filter compensates by increasing the sensitivity of the position estimates. As seen in (35), the sensitivity of posterior position to range rate innovation increases as the prior velocity variance p_{44} decreases and reaches its maximum when $p_{44} = 0$. The result is a filter whose posterior position estimates significantly depend on the range rate innovations.

6 Conclusions

The usual range rate linearization takes its partial derivatives with respect to the state’s position and velocity elements. An alternative linearization of range rate was derived. The alternative linearization takes derivatives with respect to the state’s velocity elements only, and not its position elements.

The usual and alternative linearizations were compared in extended and combined Kalman filters. The usual linearization’s functional dependence on state position was found to induce nontrivial biases into the posterior estimates. The alternative linearization does not induce large biases. Filters using the alternative linearization have more reasonable gains than those using the usual linearization. Numerical examples show that the alternative filters are more stable and less sensitive to the prior heading estimate. It was shown that in a deterministic noise-free system, for which the EKF is an exponential observer, the alternative filter converges more quickly.

Three paradoxes that are consequences of the usual linearization were discovered. First, when the prior heading estimate is cross-radial, the filter’s posterior estimates are insensitive to their respective innovations. The position innovations update the velocity estimates and the velocity innovations update the position estimate. Second, the usual linearization leads to filters whose net effective gain can be outside $[0, 1]$. That is, their posterior estimate can move from

the prior toward and beyond the measurement, or from the prior away from the measurement. Third, stronger prior beliefs can increase posterior biases. Decreasing the prior velocity variance increases the posterior position biases.

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