Web Appendix for "Equilibrium Capacity Expansion under Stochastic Demand Growth"

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1 Proof of Theorem 1 (b)

Similar with the proof of part (a), based on the initial relative capacity prior depreciation, we partition the entire relative capacity space $\mathbb{R}_+^2$ into 4 regions as figure A.1 indicates:

$$\begin{aligned} S_1 &= \{k^0 | k_i^0 \leq \tfrac{m}{3\sigma\delta}, i = 1, 2\} \\ S_2 &= \{k^0 | k_i^0 > \tfrac{m(1+g)}{3\sigma\delta}, i = 1, 2\} \\ S_3 &= \{k^0 | k_1^0 < k_2^0\} \backslash (S_1 \cup S_2) \\ S_4 &= \{k^0 | k_1^0 > k_2^0\} \backslash (S_1 \cup S_2) \end{aligned}$$

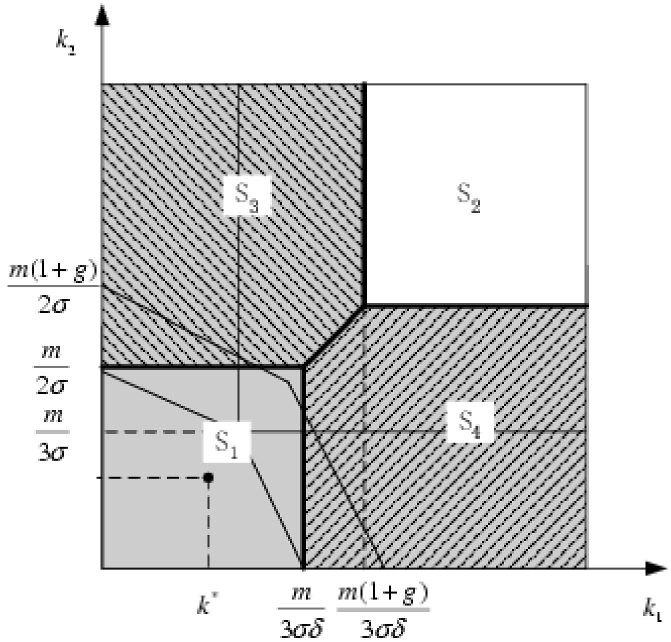

Figure A.1: The partition of relative capacity space when $\frac{m}{\kappa} \leq \frac{\rho(1+g)}{\theta g}$

We propose an investment strategy combination $\widehat{y}_1$ when $\rho < \frac{m}{\kappa} \leq \frac{\rho(1+g)}{\theta g}$, formally described as follows

$$\widehat{y}_1(k) = \begin{cases} k^* - \delta k_1 & k_i \in S_1 \\ \max\{\varphi(k) - \delta k_1; 0\} & k_i \in S_3 \\ 0 & \text{otherwise} \end{cases}$$

where

$$k^* = \frac{[\beta m - (1 - \beta\delta)\kappa](1+g)}{3\beta\sigma(1+g-\theta g)}$$

and $\varphi(k)$ is a function to be defined later. Because of symmetry, it is enough to check if the strategy $\widehat{y}_1$ is optimal for firm 1 when firm 2 follows strategy $\widehat{y}_2$. To lighten up the notation, in what follows we shall write $v_1(k)$ to refer to $v_1(k|\, y^*)$, the present value of net revenues earned by firm 1 if strategy $\widehat{y}_1$ is followed by both players upon initial condition k. We define $BR_2(k_1) = \frac{m}{2\sigma} - \frac{k_1}{2}$, $\overline{BR}_2(k_1) = \frac{m(1+g)}{2\sigma} - \frac{k_1}{2}$ when $k_1 \leq k_2$ and define $BR_1(k_2), \overline{BR}_1(k_2)$ accordingly when $k_1 > k_2$.

1.1 Region S_1

Suppose the initial relative capacities are such that $\delta k_i^0 \leq k^*$ and firm 1's capacity at the beginning of the next period after investment is $k_1 = \delta k_1^0 + y_1$. According to strategy $\widehat{y}$, firm 2's decision is to achieve a relative capacity level of k^*. So firm 1's objective function is

$$\max_{k_1}\{R_1^*(k_1^0, k_2^0) - (k_1 - \delta k_1^0)\kappa + \beta[\theta(1+g)v_1(\frac{k_1}{1+g}, \frac{k^*}{1+g}) + (1-\theta)v_1(k_1, k^*)]\}$$

The derivative $D(k_1)$ of the objective function above with respect to k_1 is:

$$D(k_1) = -\kappa + \beta\theta(1+g)\frac{\partial v_1}{\partial k_1}(\frac{k_1}{1+g}, \frac{k^*}{1+g}) + \beta(1-\theta)\frac{\partial v_1}{\partial k_1}(k_1, k^*) \tag{8.2.1}$$

If firm 1 chooses a target capacity investment level $k_1 \leq \frac{k^*}{\delta}$,

$$\begin{aligned} v_1(k_1, k^*) &= R_1^*(k_1, k^*) - (k^* - \delta k_1)\kappa + C(k^*) \\ v_1(\frac{k_1}{1+g}, \frac{k^*}{1+g}) &= R_1^*(\frac{k_1}{1+g}, \frac{k^*}{1+g}) - (k^* - \frac{\delta k_1}{1+g})\kappa + C(k^*) \end{aligned}$$

where $C(k^*) = \beta[\theta(1+g)v_1(\frac{k^*}{1+g}, \frac{k^*}{1+g}) + (1-\theta)v_1(k^*, k^*)]$ is a constant only related to the capacity level in equilibrium k^*. Thus,

$$\begin{aligned} \frac{\partial v_1}{\partial k_1}(k_1, k^*) &= \frac{\partial R_1^*}{\partial k_1}(k_1, k^*) + \delta\kappa \\ \frac{\partial v_1}{\partial k_1}(\frac{k_1}{1+g}, \frac{k^*}{1+g}) &= \frac{\partial R_1^*}{\partial k_1}(\frac{k_1}{1+g}, \frac{k^*}{1+g}) + \frac{\delta}{1+g}\kappa \end{aligned}$$

Substituting in (8.2.1) we obtain the following expression for the derivative:

$$\begin{aligned} D(k_1) &= -(1-\beta\delta)\kappa + \beta\theta(1+g)\frac{\partial R_1^*}{\partial k_1}(\frac{k_1}{1+g}, \frac{k^*}{1+g}) + \beta(1-\theta)\frac{\partial R_1^*}{\partial k_1}(k_1, k^*) \\ &= -(1-\beta\delta)\kappa + \beta\theta[m - \sigma(\frac{2k_1}{1+g} + \frac{k^*}{1+g})] + \beta(1-\theta)[m - \sigma(2k_1 + k^*)] \end{aligned}$$

Note that

$$\begin{aligned}
D(k^*) &= -(1-\beta\delta)\kappa + \beta\theta(1+g)\left.\frac{\partial R_1}{\partial k_1}\left(\frac{k_1}{1+g}, \frac{k^*}{1+g}\right)\right|_{k_1=k^*} + \beta(1-\theta)\left.\frac{\partial R_1}{\partial k_1}(k_1, k^*)\right|_{k_1=k^*} \\
&= -(1-\beta\delta)\kappa + \beta\theta\left(m - \sigma\frac{3k^*}{1+g}\right) + \beta(1-\theta)(m - \sigma 3k^*) \\
&= \beta m - (1-\beta\delta)\kappa - 3\beta\sigma k^*\left(\frac{1+g-\theta g}{1+g}\right) = 0 \\
\Rightarrow & \ k^* = \frac{[\beta m - (1-\beta\delta)\kappa](1+g)}{3\beta\sigma(1+g-\theta g)}
\end{aligned}$$

k^* is in region I only if $\rho < \frac{m}{\kappa} \leq \frac{\rho(1+g)}{\theta g}$, where $\rho = \frac{1-\beta\delta}{\beta}$.

The difference between $D(k_1)$ and $D(k^*)$ is

$$D(k_1) - D(k^*) = 2\beta\sigma\left(\frac{1+g+\theta g}{1+g}\right)(k^* - k_1)$$

We have that $D(k_1) > D(k^*)$ when $k_1 < k^*$ and $D(k_1) < D(k^*)$ for all $k^* < k_1 \leq \frac{k^*}{\delta}$.

If the initial relative capacities satisfy $k_i^0 \in S_1 \cap \{k_i^0 | k^* < \delta k_2^0 \leq k^*(1+g) \text{ and } k_2 \geq k_1\}$, and firm 1 chooses a target capacity investment level k_1, we have

$$\begin{aligned}
v_1(k_1, \delta k_2^0) &= R_1^*(k_1, \delta k_2^0) - (k^* - \delta k_1)\kappa + C(k^*) \\
v_1\left(\frac{k_1}{1+g}, \frac{\delta k_2^0}{1+g}\right) &= R_1^*\left(\frac{k_1}{1+g}, \frac{\delta k_2^0}{1+g}\right) - \left(k^* - \frac{\delta k_1}{1+g}\right)\kappa + C(k^*)
\end{aligned}$$

Similar to (8.2.1), the first order condition of firm 1's objective is:

$$\begin{aligned}
D(k_1) &= -\kappa + \beta\theta(1+g)\left[\frac{\partial R_1^*}{\partial k_1}\left(\frac{k_1}{1+g}, \frac{\delta k_2^0}{1+g}\right) + \frac{\delta\kappa}{1+g}\right] + \beta(1-\theta)\left[\frac{\partial R_1^*}{\partial k_1}(k_1, \delta k_2^0) + \delta\kappa\right] \\
&= -(1-\beta\delta)\kappa + \beta\theta(1+g)\frac{\partial R_1^*}{\partial k_1}\left(\frac{k_1}{1+g}, \frac{\delta k_2^0}{1+g}\right) + \beta(1-\theta)\frac{\partial R_1^*}{\partial k_1}(k_1, \delta k_2^0) \\
&= -(1-\beta\delta)\kappa + \beta\theta\left[m - \sigma\left(\frac{2k_1}{1+g} + \frac{\delta k_2^0}{1+g}\right)\right] + \beta(1-\theta)[m - \sigma(2k_1 + \delta k_2^0)] \\
&= 0
\end{aligned}$$

We get the investment target for firm 1 in this case

$$k_1(k_2^0) = \frac{[\beta m - (1-\beta\delta)\kappa](1+g)}{2\beta\sigma(1+g-\theta g)} - \frac{\delta k_2^0}{2} = \frac{3k^*}{2} - \frac{\delta k_2^0}{2}$$

If the initial relative capacities satisfy $k_i^0 \in S_1 \cap \{k_i^0 | k^* < \delta k_1^0 \leq k^*(1+g) \text{ and } k_2 < k_1\}$, and firm 1 chooses a target capacity investment level k_1 slightly greater than δk_1^0, we have

$$\begin{aligned} v_1(k_1, k_2(k_1^0)) &= R_1^*(k_1, k_2(k_1^0)) - (k^* - \delta k_1)\kappa + C(k^*) \\ v_1(\frac{k_1}{1+g}, \frac{k_2(k_1^0)}{1+g}) &= R_1^*(\frac{k_1}{1+g}, \frac{k_2(k_1^0)}{1+g}) - (k^* - \frac{\delta k_1}{1+g})\kappa + C(k^*) \end{aligned}$$

The derivative of firm 1's objective is:

$$\begin{aligned} D(k_1) &= -\kappa + \beta\theta(1+g)[\frac{\partial R_1^*}{\partial k_1}(\frac{k_1}{1+g}, \frac{k_2(k_1^0)}{1+g}) + \frac{\delta\kappa}{1+g}] + \beta(1-\theta)[\frac{\partial R_1^*}{\partial k_1}(k_1, k_2(k_1^0)) + \delta\kappa] \\ &= -(1-\beta\delta)\kappa + \beta\theta[m - \sigma(\frac{2k_1}{1+g} + \frac{k_2(k_1^0)}{1+g})] + \beta(1-\theta)[m - \sigma(2k_1 + k_2(k_1^0))] \end{aligned}$$

Since $k_2(k_1^0) = \frac{3k^*}{2} - \frac{\delta k_1^0}{2}$, the derivative becomes

$$\begin{aligned} D(k_1) &= -(1-\beta\delta)\kappa + \beta\theta[m - \sigma(\frac{2k_1}{1+g} + \frac{k_2(k_1^0)}{1+g})] + \beta(1-\theta)[m - \sigma(2k_1 + k_2(k_1^0))] \\ &= \beta m - (1-\beta\delta)\kappa - \beta\sigma(\frac{1+g-\theta g}{1+g})(2k_1 + \frac{3k^*}{2} - \frac{\delta k_1^0}{2}) \\ &< \beta m - (1-\beta\delta)\kappa - \beta\sigma(\frac{1+g-\theta g}{1+g})(\frac{3\delta k_1^0}{2} + \frac{3k^*}{2}) \\ &< \beta m - (1-\beta\delta)\kappa - 3\beta\sigma k^*(\frac{1+g-\theta g}{1+g}) = 0 \end{aligned}$$

So it is not profitable for firm 1 to invest in this case.

The similar analysis can be carried out for initial relative capacities in region $k_i^0 \in S_1 \cap \{k_i^0 | k^*(1+g)^t < \delta k_2^0 \leq k^*(1+g)^{t+1} \text{ and } k_2 \geq k_1\}$ and $k_i^0 \in S_1 \cap \{k_i^0 | k^*(1+g)^t < \delta k_1^0 \leq k^*(1+g)^{t+1} \text{ and } k_2 < k_1\}$ where $t = 1, 2, 3, ...$

Therefore, we conclude that for all the initial relative capacities (k_1^0, k_2^0) in region S_1, the upper envelope of $k_1(k_2^0)$ and $k_2(k_1^0)$ describes the optimal investment target for both firms. If both k_1^0 and k_2^0 are less than k^*/δ, they both invest to reach k^* at the beginning of next period; if both k_1^0 and k_2^0 are greater than the corresponding level described by $k_1(k_2^0)$ and $k_2(k_1^0)$, there will be no investment at all; for the other cases, only the firm with less relative capacity will invest to a level of capacity decided by the initial capacity of the opponent firm.

1.2 Region S_2

If $(k_1^0, k_2^0) \in \{k^0 | k_i^0 > \frac{m(1+g)}{3\sigma\delta}, i = 1, 2\}$, the relative capacities after depreciation will remain at region IV no matter whether demand growth or not. We know the short term equilibrium revenue R_1^* is a constant in region IV, so no firm has incentive to invest for this case.

1.3 Region S_3

Since we already know the optimal investment strategy $\widehat{y}_1$ in region S_1, we can identifiy the states trajectories starting from any initial relative capacities in region S_1. Therefore, the expected value $v_1(k)$ in region S_1 is known.

We divide region S_3 into a series of horizontal stripes $I_t = \{k_i^0 | k_i^0 \in S_3 \text{ and } \frac{m}{3\sigma}(1+g)^t < \delta k_2^0 \leq \frac{m}{3\sigma}(1+g)^{t+1}\}$ where $t = 0, 1, 2, ...$ If $(k_1^0, k_2^0) \in I_0$, we define firm 1's investment target as $\varphi(k_2^0)$ which is solely decided by the relative capacity level of firm 2. If firm 1 invests to reach a relative capacity level k_1, the derivative of the objective is:

$$D(k_1) = -\kappa + \beta\theta(1+g)\frac{\partial v_1}{\partial k_1}(\frac{k_1}{1+g}, \frac{\delta k_2^0}{1+g}) + \beta(1-\theta)\frac{\partial v_2}{\partial k_1}(k_1, \delta k_2^0) \qquad ((8.2.3))$$

Since there will be no investment in region S_2, firm 1's investment level is bounded by $\frac{m(1+g)}{3\sigma}$. For a fixed δk_2^0, if there is only one locally optimal k_1(that is $D(k_1) = 0$), this k_1 must be the optimal target of capacity investment for firm 1; If for example, $D(k_1) > 0$ when $k_1 < \frac{m}{3\sigma}$ and $D(k_1) < 0$ when $k_1 > \frac{m}{3\sigma}$, the optimal level of investment for firm 1 will be line $k_1 = \frac{m}{3\sigma}$; If there are more than one locally optimal level of k_1, we must compare the value of objectives corresponding to each targets levels in order to identify the real optimal level of investment level of capacity for firm 1.

After we get firm 1's investment target as $\varphi(k_2^0)$ in region I_0, we are able to calculate the expected value function of firm 1 $v_1(k)$ in this stripe. Then we can identify firm 1's investment target for $(k_1^0, k_2^0) \in I_1$. Firm 1's optimal investment target $\varphi(k_2^0)$ for those stripes with low index numbers $I_0, I_1, ...$ maybe highly discontinous along k_2^0, the reason for that is because the short term revenue is only piecewise continous in the capacity space. Let's examine $\varphi(k_2^0)$ for I_t with relatively large index number.

If firm 1 invests to reach a relative capacity level k_1, we have:

$$\begin{aligned}\frac{\partial v_1}{\partial k_1}(k_1, \delta k_2^0) &= \frac{\partial R_1^*}{\partial k_1}(k_1, \delta k_2^0) + \delta\kappa \\ \frac{\partial v_1}{\partial k_1}(\frac{k_1}{1+g}, \frac{\delta k_2^0}{1+g}) &= \frac{\partial R_1^*}{\partial k_1}(\frac{k_1}{1+g}, \frac{\delta k_2^0}{1+g}) + \frac{\delta}{1+g}\kappa\end{aligned}$$

The derivative of the objective is:

$$D(k_1) = -(1-\beta\delta)\kappa + \beta\theta(1+g)\frac{\partial R_1^*}{\partial k_1}(\frac{k_1}{1+g}, \frac{\delta k_2^0}{1+g}) + \beta(1-\theta)\frac{\partial R_1^*}{\partial k_1}(k_1, \delta k_2^0)$$

Case 1: If $(k_1, \delta k_2^0) \in II$ and $(\frac{k_1}{1+g}, \frac{\delta k_2^0}{1+g}) \in II$:, the derivative of the objective is:

$$D_1(k_1) = \frac{1}{2}\beta m - (1-\beta\delta)\kappa - \beta\sigma(\frac{\theta}{1+g} + 1 - \theta)k_1$$

Case 2: If $(k_1, \delta k_2^0) \in IV$ and $(\frac{k_1}{1+g}, \frac{\delta k_2^0}{1+g}) \in II$:

$$D_2(k_1) = \frac{1}{2}\beta m - (1-\beta\delta)\kappa - \beta\sigma(\frac{\theta}{1+g})k_1$$

Following the guidlines we just mentioned above, if there is only one local maximum, this is the optimal target investment level; if $D_1(k_1) > 0$ when $k_1 < \frac{m}{3\sigma}$ and $D_2(k_1) > 0$ when $k_1 > \frac{m}{3\sigma}$, $\varphi(k_2^0) = \frac{m(1+g)}{3\sigma}$; if $D_1(k_1) > 0$ when $k_1 < \frac{m}{3\sigma}$ and $D_2(k_1) < 0$ when $k_1 > \frac{m}{3\sigma}$, $\varphi(k_2^0) = \frac{m}{3\sigma}$; and if there are two local maxima levels, we need to compare the expected value function explicitly.

1.4 Region S_4

If the initial relative capacities (k_1^0, k_2^0) are in this region, firm 2 will invest to make its relative capacity in the next period equal to $\varphi(k_1^0)$. We are going to check if it is profitable for firm 1 to deviate by making a little investment in this region. Firm 1 will invest only if the relative capacities after depreciation and after demand growth $(\frac{\delta k_1^0}{1+g}, \frac{\varphi(k_1^0)}{1+g})$ are in region I, otherwise firm 1's one period revenue is only decided by firm 2's relative capacity. Similar with the analysis for region S_3, we divide region S_4 into a series of vertical stripes $H_t = \{k_i^0 | k_i^0 \in S_4$ and $\frac{m}{3\sigma}(1+g)^t < \delta k_1^0 \leq \frac{m}{3\sigma}(1+g)^{t+1}\}$ where $t = 0, 1, 2, ...$ For the first stripe $H_{0,}$ if $(k_1^0, k_2^0) \in H_0$ and firm 1 chooses a target level of investment $k_1 \leq \frac{m}{3\sigma\delta}$ we have:

$$v_1(k_1, \varphi(k_1^0)) = R_1^*(k_1, \varphi(k_1^0)) + \beta[\theta(1+g)v_1(\frac{k_1}{1+g}, \frac{\varphi(k_1^0)}{1+g}) + (1-\theta)v_1(k_1, \varphi(k_1^0))]$$

Thus,

$$\frac{\partial v_1}{\partial k_1}(k_1, \varphi(k_1^0)) = \frac{1}{1-\beta(1-\theta)}[\frac{\partial R_1^*}{\partial k_1}(k_1, \varphi(k_1^0)) + \beta\theta(1+g)\frac{\partial v_1}{\partial k_1}(\frac{k_1}{1+g}, \frac{\varphi(k_1^0)}{1+g})]$$

The derivative of the objective is:

$$D(k_1) = -\kappa + \frac{1}{1-\beta(1-\theta)}[\beta(1-\theta)\frac{\partial R_1^*}{\partial k_1}(k_1, \varphi(k_1^0)) + \beta\theta(1+g)\frac{\partial v_1}{\partial k_1}(\frac{k_1}{1+g}, \frac{\varphi(k_1^0)}{1+g})] \quad ((8.2.4))$$

If the relative capacities $(\frac{k_1}{1+g}, \frac{\varphi(k_1^0)}{1+g})$ are above the upper envelope of $k_1(k_2^0)$ and $k_2(k_1^0)$, firm 1 will not invest according to the optimal strategy $\widehat{y}_1$ in region S_1. So we have

$$\frac{\partial v_1}{\partial k_1}(\frac{k_1}{1+g}, \frac{\varphi(k_1^0)}{1+g}) < \frac{\kappa}{1+g}$$

The derivative in (8.2.4) becomes

$$D(k_1) < -\kappa + \frac{1}{1-\beta(1-\theta)}[\beta(1-\theta)\frac{\partial R_1^*}{\partial k_1}(k_1, \varphi(k_1^0)) + \beta\theta\kappa] \quad (i)$$

If the relative capacities $(\frac{k_1}{1+g}, \frac{\varphi(k_1^0)}{1+g})$ are under the upper envelope of $k_1(k_2^0)$ and $k_2(k_1^0)$, firm 1 will invest to $k_1(k_2^0)$ or k^* according to the optimal strategy $\widehat{y}_1$ in region S_1. So we have

$$\frac{\partial v_1}{\partial k_1}(\frac{k_1}{1+g}, \frac{\varphi(k_1^0)}{1+g}) = \frac{\partial R_1^*}{\partial k_1}(\frac{k_1}{1+g}, \frac{\varphi(k_1^0)}{1+g}) + \frac{\delta}{1+g}\kappa$$

The derivative in (8.2.4) becomes

$$D(k_1) = -(1-\beta\delta)\kappa + \beta\theta(1+g)\frac{\partial R_1^*}{\partial k_1}(\frac{k_1}{1+g}, \frac{\varphi(k_1^0)}{1+g}) + \beta(1-\theta)\frac{\partial R_1^*}{\partial k_1}(k_1, \varphi(k_1^0)) \quad (ii)$$

- **Case 1:** If the relative capacities $(k_1, \varphi(k_1^0)) \in IV$ and $(\frac{k_1}{1+g}, \frac{\varphi(k_1^0)}{1+g}) \in I$: the derivative in (i) is

$$D(k_1) < -\kappa + \frac{\beta\theta\kappa}{1-\beta(1-\theta)} = -(1-\beta) < 0$$

Since it is true for firm 2 that

$$-(1-\beta\delta)\kappa + \beta\theta[m - \frac{\sigma}{1+g}(k_1^0 + 2\varphi(k_1^0))] = 0$$

Thus,

$$\varphi(k_1^0) = \frac{(1+g)}{2\sigma}[m - \frac{(1-\beta\delta)\kappa}{\beta\theta}] - \frac{\delta k_1^0}{2}$$

and the derivative in (ii) becomes

$$\begin{aligned} D(k_1) &= -(1-\beta\delta)\kappa + \beta\theta[m - \frac{\sigma}{1+g}(2k_1 + \varphi(k_1^0))] \\ &< \frac{1}{2}[-(1-\beta\delta)\kappa + \beta\theta(m - \sigma\frac{3k_1}{1+g})] \\ &< \frac{1}{2}[-(1-\beta\delta)\kappa + \beta\theta(m - \sigma\frac{3k^*}{1+g})] = 0 \end{aligned}$$

Hence, it is not profitable for firm 1 to invest in this case.

- **Case 2:** If $(k_1, \delta k_2^0) \in III$ and $(\frac{k_1}{1+g}, \frac{\delta k_2^0}{1+g}) \in I$: the derivative in (i) is

$$D(k_1) < -\kappa + \frac{\beta\theta\kappa}{1-\beta(1-\theta)} = -(1-\beta) < 0$$

Since it is ture for firm 2 that:

$$-(1-\beta\delta)\kappa + \beta\theta[m - \frac{\sigma}{1+g}(2\varphi(k_1^0) + \delta k_1^0)] + \beta(1-\theta)\frac{1}{2}(m - 2\sigma\varphi(k_1^0)) = 0$$

Thus,

$$\varphi(k_1^0) = (\frac{1}{2 + \frac{(1-\theta)(1+g)}{\theta}})\left(\frac{[\frac{1}{2}\beta m(1+\theta) - (1-\beta\delta)\kappa](1+g)}{\beta\theta\sigma} - \delta k_1^0\right)$$

and

$$
\begin{aligned}
2k_1+\varphi(k_1^0) &> 2\delta k_1^0+\varphi(k_1^0)\\
&= \frac{2[\frac{1}{2}\beta m(1+\theta)-(1-\beta\delta)\kappa](1+g)}{\beta\theta\sigma}-2[1+\frac{(1-\theta)(1+g)}{\theta}]\varphi(k_1^0)\\
&> \frac{2[\frac{1}{2}\beta m(1+\theta)-(1-\beta\delta)\kappa](1+g)}{\beta\theta\sigma}\\
&> 3k^*
\end{aligned}
$$

Hence, the derivative in (ii) is

$$
\begin{aligned}
D(k_1) &= -(1-\beta\delta)\kappa+\beta\theta[m-\frac{\sigma}{1+g}(2k_1+\varphi(k_1^0))]\\
&< \frac{1}{2}[-(1-\beta\delta)\kappa+\beta\theta(m-\sigma\frac{3k^*}{1+g})]=0
\end{aligned}
$$

and it is not profitable for firm 1 to invest in this case.

- **Case 3:** If $(k_1,\delta k_2^0)\in I$ and $(\frac{k_1}{1+g},\frac{\delta k_2^0}{1+g})\in I$: the analysis is the same as what we did in region S_1.

  So for any initial relative capacities (k_1^0,k_2^0), there exists a MPE strategy $\widehat{y}_1$ so that any deviation from which is not profitable.

2 Uniqueness of y^*

Consider the class of investment strategies, say $\tilde{y}(k)$, that prescribe investments $\tilde{y}_i(k)=\tilde{k}_i-\delta k_i$ whenever $\delta k_i\le\tilde{k}_i$ and $\tilde{k}_1\ne\tilde{k}_2$. Suppose the initial capacity stock k^0 is such that $\delta k_i^0\le\tilde{k}_i$. Firm i's objective function is

$$
\max_{k_i}\{R_i^*(k^0)-(k_i-\delta k_i^0)\kappa+\beta[\theta(1+g)v_i(\frac{k_i}{1+g},\frac{\tilde{k}_j}{1+g}\Big|\tilde{y})+(1-\theta)v_i(k_i,\tilde{k}_j\Big|\tilde{y})]\}
$$

The (necessary) first order condition for equilibrium is for $i\in\{1,2\}$:

$$
(1-\beta\delta)\kappa=\beta\theta(1+g)\frac{\partial R_i^*}{\partial k_i}(\frac{\tilde{k}_i}{1+g},\frac{\tilde{k}_j}{1+g})+\beta(1-\theta)\frac{\partial R_i^*}{\partial k_i}(\tilde{k}_i,\tilde{k}_j)
$$

Without loss of generality, it is enough to consider the condition where $\tilde{k}_i \leq \tilde{k}_j$. The marginal profits are as follows:

- Region I:

$$\begin{cases} \frac{\partial R_i^*}{\partial k_i}(\tilde{k}_i, \tilde{k}_j) = m - \sigma(2\tilde{k}_i + \tilde{k}_j) \\ \frac{\partial R_i^*}{\partial k_j}(\tilde{k}_i, \tilde{k}_j) = m - \sigma(\tilde{k}_i + 2\tilde{k}_j) \end{cases}$$

- Region II:

$$\begin{cases} \frac{\partial R_i^*}{\partial k_i}(\tilde{k}_i, \tilde{k}_j) = \frac{m}{2} - \sigma\tilde{k}_i \\ \frac{\partial R_i^*}{\partial k_j}(\tilde{k}_i, \tilde{k}_j) = 0 \end{cases}$$

- Region IV:

$$\begin{cases} \frac{\partial R_i^*}{\partial k_i}(\tilde{k}_i, \tilde{k}_j) = 0 \\ \frac{\partial R_i^*}{\partial k_j}(\tilde{k}_i, \tilde{k}_j) = 0 \end{cases}$$

Note that best reply functions for firm i and j are linear with different slopes when $(\tilde{k}_i, \tilde{k}_j)$ and $(\frac{\tilde{k}_i}{1+g}, \frac{\tilde{k}_j}{1+g}) \in \{I, IV\}$. So there can only exists a symmetric equilibrium in these cases. If $(\tilde{k}_i, \tilde{k}_j) \in II$ and $(\frac{\tilde{k}_i}{1+g}, \frac{\tilde{k}_j}{1+g}) \in I$. The best reply of firm i is:

$$\tilde{k}_j^{(i)} = \frac{1+g}{\beta\theta\sigma}[\frac{1}{2}\beta\theta(1+g) - (1-\beta\delta)\kappa - \beta\sigma(\frac{2\theta}{1+g} + 1 - \theta)\tilde{k}_i]$$

The best reply of firm j is:

$$\tilde{k}_j^{(j)} = (m - \frac{\rho\kappa}{\theta})\frac{1+g}{2\sigma} - \frac{\tilde{k}_i}{2}$$

The slope of firm $i's$ best reply equals

$$-\frac{1+g}{\theta}(\frac{2\theta}{1+g} + 1 - \theta) = -2 - \frac{(1+g)(1-\theta)}{\theta} \leq -2$$

which is steeper than firm j's best reply. We check for an intersection of best reply maps when $(\tilde{k}_i, \tilde{k}_j) \in II$ and $(\frac{\tilde{k}_i}{1+g}, \frac{\tilde{k}_j}{1+g}) \in I$. When $\tilde{k}_i = \frac{m}{3\sigma}$, $\tilde{k}_i$ locates at the right boundary of this region, the corresponding $\tilde{k}_j$ from firm $i's$ best reply is

$$\tilde{k}_j^{(i)} = \frac{m}{6\theta\sigma}(1 + g + \theta + 5\theta g) - \frac{(1+g)\rho\kappa}{\theta\sigma}$$

The corresponding $\tilde{k}_j$ from firm $j's$ best reply is

$$\tilde{k}_j^{(j)} = \frac{m}{6\sigma}(2+3g) - \frac{(1+g)\rho\kappa}{2\theta\sigma}$$

Hence, the difference is:

$$\tilde{k}_j^{(i)} - \tilde{k}_j^{(j)} = \frac{m}{6\theta\sigma}(1+g-\theta+2\theta g) - \frac{(1+g)\rho\kappa}{2\theta\sigma}$$

Remember that we require $\frac{m}{\kappa} > \frac{\rho(1+g)}{\theta g}$ in Theorem 1, we get $\kappa < \frac{m\theta g}{\rho(1+g)}$. Plug this relation into the above expression, this leads to

$$\begin{aligned}
\tilde{k}_j^{(i)} - \tilde{k}_j^{(j)} &> \frac{m}{6\theta\sigma}(1+g-\theta+2\theta g) - \frac{m\theta g}{2\theta\sigma} \\
&= \frac{m}{6\theta\sigma}(1+g)(1-\theta) \geq 0
\end{aligned}$$

Therefore, firm i's best reply is always above firm j's best reply, so these two best replies will not intersect in this region. If $((\tilde{k}_i, \tilde{k}_j) \in II$ and $(\frac{\tilde{k}_i}{1+g}, \frac{\tilde{k}_j}{1+g}) \in II)$or $((\tilde{k}_i, \tilde{k}_j) \in IV$ and $(\frac{\tilde{k}_i}{1+g}, \frac{\tilde{k}_j}{1+g}) \in II)$, we have that the right side of first order condition for equilibrium for firm j is always equal to 0. This means firm j has no incentive to invest to these regions.